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Decay Estimates for Resolvents of Volterra Equations

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1. INTRODUCTION AND STATEMENT OF RESULTS

The purpose of this paper is to establish some estimates on the decay (as $t \to \infty$) of the resolvents of the (scalar) Volterra integral and integrodifferential equations

$$x(t) + \int_0^t a(t-s)\,x(s)\,ds = f(t), \qquad t \in R^+ = [0, \infty) \tag{1.1}$$

and

$$y'(t) + A_0 y(t) + \int_0^t A(t-s)\,y(s)\,ds = F(t), \qquad t \in R^+, y(0) = y_0, \tag{1.2}$$

where prime ($'$) denotes differentiation. The resolvents r and R are defined to be the solutions of the equations

$$r(t) + \int_0^t a(t-s)\,r(s)\,ds = a(t), \qquad t \in R^+, \tag{1.3}$$

and

$$R'(t) + A_0 R(t) + \int_0^t A(t-s)\,R(s)\,ds = 0, \qquad t \in R^+, R(0) = 1. \tag{1.4}$$

Observe that if $A_0 + \int_0^t A(s)\,ds = a(t)$, then $R'(t) = -r(t)$ and see, e.g., [1–10] and [12] for earlier results on the resolvents of (1.1) and (1.2).

Since the solutions of (1.1) and (1.2) are given by

$$x(t) = f(t) - \int_0^t r(t-s) f(s)\,ds, \qquad t \in R^+, \tag{1.5}$$

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and

$$y(t) = y_0 R(t) + \int_0^t R(t-s) F(s)\, ds, \qquad t \in R^+, \tag{1.6}$$

it is clear why one is interested in the asymptotic behavior of r and R as $t \to \infty$. Thus one sees for example that if $f \in C_0(R^+)$ and $r \in L^1(R^+)$, then $x \in C_0(R^+)$, but knowledge of the decay rate of f is of no use unless one knows something about the decay rates of the functions $r(t)$ or $\int_t^\infty |r(s)|\, ds$. The same observation holds for Eq. (1.2) and the resolvent R. For a certain class of kernels a and A, (essentially convex, nonincreasing functions) the theorems below give decay estimates for r and R that combined with knowledge about f and F give information about the asymptotic behavior (as $t \to \infty$) of the solutions x and y of (1.1) and (1.2).

We let $\hat{}$ denote the Fourier transform, $\hat{g}(z) = \int_0^\infty e^{-izt} g(t)\, dt$ if g is defined on R^+.

THEOREM 1. *Assume that*

$$a = \alpha a_1 + a_2, \qquad \alpha \in \mathbb{C}, |\alpha| = 1, \tag{1.7}$$

where

$$a_1 \in L^1_{\text{loc}}(R^+) \text{ is nonnegative, nonincreasing and convex on } (0, \infty), \tag{1.8}$$

$$\lim_{t \to \infty} a_1(t) = 0 \qquad \text{and} \qquad \int_0^\infty a_1(t)\, dt = +\infty, \tag{1.9}$$

$$\begin{gathered} a_2 \in AC_{\text{loc}}((0, \infty)), \lim_{t \to \infty} a_2(t) = \lim_{t \to \infty} a_2'(t) = 0 \\ \text{and } a_3, a_4 \in L^1_{\text{loc}}(R^+), \text{ where } a_3(t) = \int_t^\infty |a_2'(s)|\, ds, \\ a_4(t) = \int_t^\infty \operatorname{var}(a_2' \colon [s, \infty))\, ds, \end{gathered} \tag{1.10}$$

$$\begin{gathered} \limsup_{t \to \infty} \left(\int_0^t a_3(s)\, ds \right) \left(\int_0^t a_1(s)\, ds \right)^{-1} < 2^{-3/2}, \\ \sup_{t > 1} \left(\int_0^t s^j a_4(s)\, ds \right) \left(\int_0^t s^j a_1(s)\, ds \right)^{-1} < \infty, \qquad j = 0, 1, \end{gathered} \tag{1.11}$$

$$1 + \hat{a}(z) \neq 0, \operatorname{Im} z \leqslant 0, \tag{1.12}$$

$$r \text{ is the solution of } (1.3). \tag{1.13}$$

Then

$$\int_T^\infty |r(t)|\,dt = O\left(T^{-1}\int_0^T t^{-1}\left(1+\int_0^t a_1(s)\,ds\right)\right.$$
$$\left.\times\int_0^t\left(1+\int_0^s a_1(u)\,du\right)^{-2} ds\,dt\right) \qquad as \quad T\to\infty \quad (1.14)$$

and

$$r(T) = O\left(T^{-2}\left(1+\int_0^T a_1(t)\,dt\right)\int_0^T\left(1+\int_0^t a_1(s)\,ds\right)^{-2} dt\right). \quad (1.15)$$

The main reason for the assumption that $\lim_{t\to\infty} a(t)=0$ is that if this is not the case, then one could apply known results (see [4] and [10]) that imply $\int_0^\infty |r(t)|\, m(t)\,dt<\infty$ (where m is a nondecreasing continuous function such that $m(t+s)\leqslant m(t)\,m(s)$ and $m(0)=1$), so that $\int_T^\infty |r(t)|\,dt = o((m(t))^{-1})$ as $t\to\infty$. This approach of using weighted spaces seems also to be the best one in the case when a is integrable and no convexity assumptions are needed (see [10]).

The fact that $r\in L^1(R^+)$ if the assumptions of Theorem 1 hold does not follow from the results in, e.g., [2, 8 or 12]. Note also that if the function a in Theorem 1 is a real function, then it is the difference of two, nonnegative, nonincreasing and convex functions. In any case, the functions a_3 and a_4 are nonnegative and nonincreasing, a_4 is convex and $|a_2(t)|\leqslant a_3(t)\leqslant a_4(t)$, $t\in R^+$.

If $a(t)=t^{-q}$, $q\in(0,1)$, then $r(t)=O(t^{q-2})$ as $t\to\infty$ and one sees that (1.15) gives this estimate only in the case when $q\in(2^{-1},1)$. One can, however, obtain better estimates than those in (1.14) and (1.15), provided that the monotonicity assumptions on a_1 are strengthed.

THEOREM 2. *Assume that* (1.7)–(1.10), (1.12) *and* (1.13) *hold and that*

$$-a_1' \text{ is convex on } (0,\infty), \qquad (1.16)$$

$$a_4\in L^1(R^+). \qquad (1.17)$$

Then

$$\int_T^\infty |r(t)|\,dt = O\left(T^{-1}\int_0^T t^{-1}\int_0^t\left(1+\int_0^s a_1(u)\,du\right)^{-1} ds\,dt\right)$$
$$as \quad T\to\infty \quad (1.18)$$

and

$$r(T) = O\left(T^{-2}\int_0^T \left(1+\int_0^t a_1(s)\,ds\right)^{-1} dt\right) \qquad as \quad T\to\infty. \tag{1.19}$$

This theorem extends some results in [15] where it is assumed that $\liminf_{t\to\infty} t^{-q}\int_0^t a_1(s)\,ds > 0$, $q\in(0,1)$.

Concerning the resolvent R of Eq. (1.2) we have the following versions of Theorems 1 and 2. These results are established through a reduction of Eq. (1.4) to the form (1.3), cf. [11].

THEOREM 3. *Assume that*

$$A_0 \in \mathbb{C}, \tag{1.20}$$

$$A = \alpha a_1 + a_2,\ \alpha\in\mathbb{C},\ |\alpha| = 1, \textit{ where } a_1 \textit{ and } a_2 \textit{ satisfy } (1.8)\text{–}(1.11), \tag{1.21}$$

$$iz + A_0 + A\hat{}(z) \neq 0, \qquad \operatorname{Im} z \leqslant 0, \tag{1.22}$$

$$R \textit{ is the solution of } (1.4). \tag{1.23}$$

Then (1.14) *and* (1.15) *hold with r replaced by R.*

THEOREM 4. *Assume that* (1.16), (1.17) *and* (1.20)–(1.23) *hold. Then* (1.18) *and* (1.19) *hold with r replaced by R,*

2. PROOF OF THEOREMS 1 AND 2

First we establish some useful facts about the Fourier transform of a. From results in [12, p. 320] and (1.8)–(1.10) we conclude that $a\hat{}(z)$ is continuously differentiable in $\operatorname{Im} z\leqslant 0$, $z\neq 0$, and the following crucial estimates hold

$$|\hat{a}_1'(x)| \leqslant 40\int_0^{1/|x|} t a_1(t)\,dt, \qquad |x|\neq 0 \tag{2.1}$$

$$8^{-1/2}\int_0^{1/|x|} a_1(t)\,dt \leqslant |\hat{a}_1(x)| \leqslant 4\int_0^{1/|x|} a_1(t)\,dt, \qquad x\neq 0 \tag{2.2}$$

$$\left.\begin{aligned} |\hat{a}_2(x)| &\leqslant 4\int_0^{1/|x|} a_4(t)\,dt, \\ |\hat{a}_2'(x)| &\leqslant 40\int_0^{1/|x|} t a_4(t)\,dt, \end{aligned}\right\} \qquad x\neq 0. \tag{2.3}$$

By (1.3) and (1.12) we deduce that $r\hat{}(z)$ is continuously differentiable in $\operatorname{Im} z \leqslant 0$, $z \neq 0$ and we have

$$r\hat{}'(x) = a\hat{}'(x)(1 + a\hat{}(x))^{-2}, \qquad x \neq 0. \tag{2.4}$$

Next we want to show that there exists a positive constant c_1 such that

$$|1 + a\hat{}(z)| \geqslant c_1 \max\{1, |a_1\hat{}(z)|\}, \qquad \operatorname{Im} z \leqslant 0, z \neq 0. \tag{2.5}$$

Since (1.9), (1.12) and (2.2) hold it is sufficient to show that

$$\limsup_{z \to 0, \operatorname{Im} z \leqslant 0} |a_2\hat{}(z)|/|a_1\hat{}(z)| < 1. \tag{2.6}$$

It follows from results in [12, p. 320] and from the convexity of a_1 that

$$\begin{aligned} |a_1\hat{}(x+iy)| &\geqslant 2^{-1/2} \int_0^{\pi/|2x|} \cos(xt)\, e^{yt} a_1(t)\, dt \\ &= 2^{-1/2} \int_0^{\pi/|2x|} (x \sin(xt) - y\cos(xt))\, e^{yt} \int_0^t a_1(s)\, ds\, dt, \\ &\qquad\qquad y \leqslant 0,\ x \neq 0. \end{aligned} \tag{2.7}$$

Applying [12, line (1.6)] we obtain after some integrations by parts (see (1.10))

$$\begin{aligned} |a_2\hat{}(x+iy)| &\leqslant |x|^{-1} \int_0^{\infty} |1 - e^{-ixt}|\, e^{yt}\, |ya_2(t) + a_2'(t)|\, dt \\ &\leqslant -2\,|x|^{-1} \int_0^{\pi/|2x|} |\sin(xt)|\, d/dt(e^{yt} a_3(t))\, dt \\ &\quad - 2\,|x|^{-1} \int_{\pi/|2x|}^{\infty} d/dt(e^{yt} a_3(t))\, dt \\ &= 2 \int_0^{\pi/|2x|} (x\sin(xt) - y\cos(xt))\, e^{yt} \int_0^t a_3(s)\, ds\, dt, \\ &\qquad\qquad y \leqslant 0,\ x \neq 0, \end{aligned}$$

This inequality combined with (1.11), (2.7) and the fact that $\int_0^\infty a_1(s)\, ds = +\infty$ yields (2.6).

The following lemma will be the key to the proof of Theorem 1.

LEMMA 2.1. *If the assumptions of Theorem* 1 *hold, then* $\hat{r}'(x) \in L^1(R^+)$ *and*

$$\int_{-\infty}^{\infty} |\hat{r}'(x) - \hat{r}'(x+u)|\, dx$$

$$= O\left(|u| \left(1 + \int_0^{1/|u|} a_1(t)\, dt\right) \int_0^{1/|u|} \left(1 + \int_0^t a_1(s)\, ds\right)^{-2} dt\right),$$

$$u \to 0, \ u \text{ real}. \qquad (2.8)$$

Proof. The fact that $\hat{r}'(x) \in L^1(R^+)$ will follow from the same arguments that are used to establish (2.8) so will concentrate on that relation.

We need the fact, almost established in [12, pp. 322, 323], that if the function w is defined by

$$w(x) = \int_0^{1/x} t a_1(t)\, dt \left(\int_0^{1/x} a_1(t)\, dt\right)^{-2}, \qquad x > 0 \qquad (2.9)$$

then w is nonincreasing and

$$\int_0^x w(t)\, dt \leqslant 2 \left(\int_0^{1/x} a_1(t)\, dt\right)^{-1}, \qquad x > 0. \qquad (2.10)$$

To see this, note that in [12, line (1.21)] one should have $\delta\psi(1/\delta)$ on the right-hand side and then an obvious estimate and an evaluation of the integral there gives (2.10).

Let $u \in (0, 1)$. Clearly

$$\int_{-2u}^{u} |\hat{r}'(x) - \hat{r}'(x+u)|\, dx \leqslant 2 \int_{-u}^{u} |\hat{r}'(x)|\, dx + \int_{-2u}^{-u} |\hat{r}'(x)|\, dx$$

$$+ \int_{u}^{2u} |\hat{r}'(x)|\, dx. \qquad (2.11)$$

From (1.7), (2.1), (2.2), (2.4), (2.5) and (2.9) we conclude that there exists a constant c_2 so that

$$|\hat{r}'(x)| \leqslant c_2 w(|x|), \qquad |x| \leqslant 2,$$

since by (1.11) and (2.2) $\sup_{|x| \leqslant 2} |\hat{a}_2'(x)| (\int_0^{1/|x|} t a_1(t)\, dt)^{-1} < \infty$. Therefore it follows from (2.10), (2.11) and the fact that w is nonincreasing that

$$\int_{-2u}^{u} |\hat{r}'(x) - \hat{r}'(x+u)|\, dx \leqslant 12 c_2 \left(\int_0^{1/u} a_1(t)\, dt\right)^{-1}. \qquad (2.12)$$

Next we observe that by (2.4)

$$\begin{aligned} \hat{r}'(x) - \hat{r}'(x+u) &= \hat{a}'(x+u)(\hat{a}(x+u) - \hat{a}(x))(2 + \hat{a}(x) \\ &\quad + \hat{a}(x+u))(1 + \hat{a}(x))^{-2}(1 + \hat{a}(x+u))^{-2} \\ &\quad + (\hat{a}'(x) - \hat{a}'(x+u))(1 + \hat{a}(x))^{-2} \\ &\stackrel{\text{def}}{=} q_1(x, u) + q_2(x, u). \end{aligned} \tag{2.13}$$

We may, without loss of generality, assume that $x > 0$, since the case when $x < 0$ can be treated in a similar manner. From (1.7), (2.1) and (2.3) we deduce that

$$|\hat{a}'(x+u)| \leqslant 40 \int_0^{1/(x+u)} t(a_1(t) + a_4(t))\, dt, \tag{2.14}$$

$$|\hat{a}(x+u) - \hat{a}(u)| \leqslant 40u \int_0^{1/x} t(a_1(t) + a_4(t))\, dt \tag{2.15}$$

and from (1.7), (2.2) and (2.3) and the fact that a_1 and a_4 are nonincreasing we see that

$$\begin{aligned} |\hat{a}(x) + \hat{a}(x+u)| &\leqslant 16 \int_0^{1/(x+u)} (a_1(t) + a_4(t))\, dt \qquad \text{if} \quad x \in (0, 1), \\ &\leqslant 8 \int_0^1 (a_1(t) + a_4(t))\, dt \qquad \text{if} \quad x \geqslant 1. \end{aligned} \tag{2.16}$$

It follows from (1.11), (2.2), (2.5) and (2.13)–(2.16) (since $\int_0^1 a_1(s)\, ds > 0$) that there exists a constant c_3 such that

$$\begin{aligned} |q_1(x, u)| &\leqslant c_3 u \left(\int_0^{1/(x+u)} t a_1(t)\, dt\right)\left(\int_0^{1/x} t a_1(t)\, dt\right) \\ &\left(\int_0^{1/x} a_1(t)\, dt\right)^{-2}\left(\int_0^{1/(x+u)} a_1(t)\, dt\right)^{-1} \leqslant 2c_3 u x^{-1} w(x), \qquad x \in (0, 1) \end{aligned} \tag{2.17}$$

and because a_1 and $a_4 \in L^1_{\text{loc}}(R^+)$

$$|q_1(x, u)| \leqslant c_4 u x^{-1} \int_0^{1/x} t(a_1(t) + a_4(t))\, dt, \qquad x \geqslant 1, \tag{2.18}$$

for some constant c_4.

An easy calculation involving (2.10) gives

$$\int_0^1 x^{-1}w(x)\,dx \leqslant 2\left(\int_0^1 a_1(t)\,dt\right)^{-1} + 2\int_1^{1/u}\left(\int_0^x a_1(t)\,dt\right)^{-1} dx. \tag{2.19}$$

Moreover, we have

$$\int_1^\infty x^{-1}\int_0^{1/x} t(a_1(t)+a_4(t))\,dt\,dx = \int_0^1 x^{-1}\int_0^x t(a_1(t)+a_4(t))\,dt < \infty \tag{2.20}$$

Thus we conclude from (2.17)–(2.20) that

$$\int_u^\infty |q_1(x,u)|\,dx = O\left(u\int_0^{1/u}\left(1+\int_0^x a_1(t)\,dt\right)^{-1} dx\right) \quad \text{as} \quad u \to 0+. \tag{2.21}$$

We proceed to consider the term $q_2(x, u)$ in (2.13) and we have the following expression for $\hat{a_j}'(x)$, $j = 1, 2$, see [12, p. 320],

$$\hat{a_j}'(x) = -\int_{R+} H(x,t)\,da_j'(t), \qquad j = 1, 2, \tag{2.22}$$

where

$$H(x,t) = x^{-3}(2(e^{-ixt} + ixt - 1) + ixt(e^{-ixt} - 1)).$$

A calculation shows that

$$|H(x,t) - H(x+u,t)| \leqslant 3/4ux^{-1}t^3, \qquad xt < 1, x > 0. \tag{2.23}$$

When $xt \geqslant 1$ we have

$$|H(x,t) - H(x+u,t)| \leqslant 10ux^{-4}(xt+2) + \begin{cases} 2x^{-2}t, & ut \geqslant 1, \\ ux^{-2}t^2, & ut \leqslant 1. \end{cases} \tag{2.24}$$

Now

$$\int_{[0,1/x)} t^3\,da_1'(t) + \int_{[0,1/x)} t^3\,|da_2'(t)| \leqslant \int_0^{1/x} t(a_1(t)+a_4(t))\,dt \tag{2.25}$$

and

$$\int_{[1/x,\infty)} (xt+2)\,da_1'(t) + \int_{[1/x,\infty)} (xt+2)\,|da_2'(t)|$$
$$\leqslant 18x^3 \int_0^{1/x} t(a_1(t)+a_4(t))\,dt, \tag{2.26}$$

(for more details, see [12, p. 321]). By similar arguments we get

$$\int_{[1/x,1/u)} t^2\,da_1'(t) + \int_{[1/x,1/u)} t^2\,|da_2'(t)|$$
$$\leqslant 6x \int_0^{1/x} t(a_1(t)+a_4(t))\,dt + 2\int_{1/x}^{1/u} (a_1(t)+a_4(t))\,dt \tag{2.27}$$

and

$$\int_{[1/u,\infty)} t\,da_1'(t) + \int_{[1/u,\infty)} t\,|da_2'(t)|$$
$$\leqslant 6u^2 \int_0^{1/u} t(a_1(t)+a_4(t))\,dt. \tag{2.28}$$

From (1.11), (2.2), (2.5) and (2.22)–(2.28) combined with the definition of $q_2(x,u)$ we get

$$\begin{aligned}
&\int_u^1 |q_2(x,u)|\,dx \\
&\quad\leqslant c_5 u \left(\int_u^1 x^{-1} w(x)\,dx\right. \\
&\qquad + \int_0^1 x^{-2} \int_{1/x}^{1/u} (a_1(t)+a_4(t))\,dt \left(\int_0^{1/x} a_1(t)\,dt\right)^{-2} dx \\
&\qquad \left. + u\int_u^1 x^{-2} \left(\int_0^{1/u} t a_1(t)\,dt\right)\left(\int_0^{1/x} a_1(t)\,dt\right)^{-2} dx\right) \\
&\quad = O\left(u \int_0^{1/u} a_1(t)\,dt \int_0^{1/u} \left(1+\int_0^t a_1(s)\,ds\right)^{-2} dt\right) \\
&\qquad\qquad\qquad\qquad \text{as} \quad u\to 0,
\end{aligned} \tag{2.29}$$

because (2.19) holds. If we once more invoke (2.2), (2.5) and (2.22)–(2.28), then we get

$$\begin{aligned}
&\int_1^{\infty} |q_2(x,u)|\,dx \\
&\quad \leqslant c_6 u \left(\int_1^{\infty} x^{-1}\int_0^{1/x} t(a_1(t)+a_4(t))\,dt\,dx\right. \\
&\qquad + \int_1^{\infty} x^{-2}\int_{1/x}^{1/u} (a_1(t)+a_4(t))\,dt\,dx \\
&\qquad \left. + u\int_0^{1/u} t(a_1(t)+a_4(t))\,dt \int_1^{\infty} x^{-2}\,dx\right) \\
&\quad = O\left(u\left(1+\int_0^{1/u} a_1(s)\,ds\right)\int_0^{1/u}\left(1+\int_0^t a_1(s)\,ds\right)^{-2} dt\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as} \quad u\to 0, \qquad (2.30)
\end{aligned}$$

because (1.11) holds and

$$\begin{aligned}
&u\left(1+\int_0^1 a_1(t)\,dt\right)^{-1} \\
&\quad \leqslant u\int_0^{1/u}\left(1+\int_0^t a_1(s)\,ds\right)^{-1} dt \\
&\quad \leqslant u\left(1+\int_0^{1/u} a_1(s)\,ds\right)\int_0^{1/u}\left(1+\int_0^t a_1(s)\,ds\right)^{-2} dt
\end{aligned}$$

and

$$\begin{aligned}
&u\int_0^{1/u} a_1(s)\,ds \\
&\quad = O\left(u\left(1+\int_0^{1/u} a_1(s)\,ds\right)\int_0^{1/u}\left(1+\int_0^t a_1(s)\,ds\right)^{-2} dt\right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as} \quad u\to 0.
\end{aligned}$$

Since the integral of $|\hat{r}'(x)-\hat{r}'(x+u)|$ over $(-\infty,-2u)$ gives the same upper bound as that over (u,∞) and since no generality was lost in assuming that $u>0$, we see that (2.8) follows from (2.12), (2.21), (2.29) and (2.30). Thus the proof of Lemma 2.1 is completed.

Proceeding in the same way as in [12, p. 323], (use $\hat{r}'(x)\in L^1(R)$ and (2.5)), we see that $r\in L^1(R^+)$ and by the Fourier inversion formula we have

$$\begin{aligned}
&r(t)(-2\pi i t)(i-e^{-ic}) \\
&\quad = \int_{-\infty}^{\infty} e^{ixt}(\hat{r}'(x)-\hat{r}'(x+ct^{-1}))\,dx, \qquad t>0. \qquad (2.31)
\end{aligned}$$

We conclude that (1.15) follows from (2.8), (take $c=1$).

It is well known that if one defines

$$p(t) = \pi^{-1} t^{-2}(\cos(t) - \cos(2t)),$$
$$p_T(t) = Tp(Tt),\ t \in (-\infty, \infty),\ T > 0, \tag{2.32}$$

then

$$\hat{p_T}(x) = \max\{0, \min\{2 - T^{-1}|x|, 1\}\}, \qquad x \in (-\infty, \infty).$$

We let

$$h_T(t) = (1 - \hat{p_T}(t))\, r(t), \qquad t \geqslant 0,\ h_T(t) = 0,\ t < 0 \tag{2.33}$$

so that

$$\hat{h_T}(x) = \hat{r}(x) - \int_{-\infty}^{\infty} \hat{r}(x-u)\, p_T(u)\, du, \qquad x \in (-\infty, \infty). \tag{2.34}$$

Since $r \in L^1(R^+)$ we have $h_T \in L^1(R^+)$, and therefore we are able to conclude in the same way as in [12, pp. 323, 324] that, (see (2.32)–(2.34)),

$$\int_T^{\infty} |r(t)|\, dt \leqslant \int_0^{\infty} |h_T(t)|\, dt \leqslant 2^{-1} \int_{-\infty}^{\infty} |\hat{h_T}'(x)|\, dx \tag{2.35}$$

$$\leqslant 2^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{r}'(x) - \hat{r}'(x-u)|\, dx\, |p_T(u)|\, du, \qquad T > 0$$

Let

$$g(u) = \int_{-\infty}^{\infty} |\hat{r}'(x) - \hat{r}'(x-u)|\, dx, \qquad u \in (-\infty, \infty). \tag{2.36}$$

This function g is by Lemma 2.1 bounded and therefore we have by (2.32) for some constant c_7

$$\int_{-\infty}^{\infty} g(u)\, |p_T(u)|\, du \leqslant c_7 \left(\int_{-c}^{c} g(u/T) + \int_c^{cT} g(u/T)\, u^{-2}\, du \right.$$
$$\left. + \int_{-cT}^{-c} g(u/T)\, u^{-2}\, du + (cT)^{-1} \right), \tag{2.37}$$

where $c > 0$ is a constant, (here we will use $c = 1$ but we need $c \neq 1$ in the proofs of Theorems 3 and 4). We have

$$\int_c^{cT} (g(u/T) + g(-u/T))\, u^{-2}\, du$$
$$= T^{-1} \int_{1/c}^{T/c} (g(1/u) + g(-1/u))\, du. \tag{2.38}$$

Since (2.8) and (2.36) hold,

$$\int_{-c}^{c} g(u/T)\,du = O\left((T/c)^{-5/3}\left(1+\int_0^{T/c} a_1(s)\,ds\right)\right.$$
$$\times \int_0^{T/c}\left(1+\int_0^t a_1(s)\,ds\right)^{-2} dt$$
$$\left.\times\left(\int_0^c (u/T)^{-2/3}\,du\right)\right) \qquad \text{as} \quad T\to\infty, \tag{2.39}$$

provided that we can show that the function $v(t) =^{\text{def}} t^{-5/3}(1+\int_0^t a_1(s)\,ds)$ $\int_0^t (1+\int_0^s a_1(u)\,du)^{-2}\,ds$ is nonincreasing. Since $(1+\int_0^t a_1(s)\,ds)^{-2}$ is a convex function by (1.8), we have

$$\int_0^t\left(1+\int_0^s a_1(u)\,du\right)^{-2} ds \geqslant t\left(1+\int_0^t a_1(s)\,ds\right)^{-2}$$
$$+\,t^2 a_1(t)\left(1+\int_0^t a_1(s)\,ds\right)^{-3}, \qquad t>0$$

and therefore it follows that

$$v'(t) \leqslant t^{-5/3}\left(1+\int_0^t a_1(s)\,ds\right)^{-2}\left(\left(-5/3t^{-1}\left(1+\int_0^t a_1(s)\,ds\right)+a_1(t)\right)\right.$$
$$\left.\times\left(t+t^2a_1(t)\left(1+\int_0^t a_1(s)\,ds\right)^{-1}\right)+\left(1+\int_0^t a_1(s)\,ds\right)\right)$$
$$= t^{-5/3}\left(1+\int_0^t a_1(s)\,ds\right)^{-1}\left(\left(-5/3+ta_1(t)\left(1+\int_0^t a_1(s)\,ds\right)^{-1}\right)\right.$$
$$\left.\times\left(1+ta_1(t)\left(1+\int_0^t a_1(s)\,ds\right)^{-1}\right)+1\right)\leqslant 0$$

because $ta_1(t) \leqslant 1+\int_0^t a_1(s)\,ds$ by (1.8).

If we take $c=1$ and use (2.8) and (2.35)–(2.39) then we obtain (1.14) since applying the same argument that was used in deriving (2.38) one gets

$$T^{-1}\left(1+\int_0^T a_1(s)\,ds\right)\int_0^T\left(1+\int_0^t a_1(s)\,ds\right)^{-2} dt$$
$$= O\left(T^{-1}\int_0^T t^{-1}\left(1+\int_0^t a_1(s)\,ds\right)\right.$$
$$\left.\times\int_0^t\left(1+\int_0^s a_1(u)\,du\right)^{-2} ds\,dt\right) \qquad \text{as} \quad T\to\infty.$$

This completes the proof of Theorem 1.

We proceed to the proof of Theorem 2 and we observe that it will be sufficient to establish the following improved version of the relation (2.8)

$$\int_{-\infty}^{\infty} |\hat{r}'(x) - \hat{r}'(x+u)|\, dx$$
$$= O\left(|u| \int_0^{1/|u|} \left(1 + \int_0^t a_1(s)\, ds\right)^{-1} dt\right) \qquad \text{as} \quad u \to 0. \quad (2.40)$$

We see from the proof of Lemma 2.1 that the only term that did not give this estimate was $\int_0^\infty |q_2(x, u)|\, dx$, $(u > 0)$. But since (1.16) holds we know that $\hat{a_1}'(z)$ is continuously differentiable in $\operatorname{Im} z \leqslant 0$, $z \neq 0$ and

$$|\hat{a_1}''(x)| \leqslant 6000 \int_0^{1/|x|} t^2 a_1(t)\, dt, \qquad x \neq 0,$$

see [1, p. 972]. But then, if $0 < u < x$,

$$|a_1'(x) - a_1'(x+u)| \leqslant 6000 u x^{-1} \int_0^{1/x} t a_1(t)\, dt.$$

From (1.17) and (2.22)–(2.28) we conclude that

$$|\hat{a_2}'(x) - \hat{a_2}'(x+u)| \leqslant c_8 u x^{-2}, \qquad 0 < u < x,$$

for some constant c_8 and having these two inequalities we can proceed in the same way as in the proof of Lemma 2.1 to derive (2.40). This completes the proof of Theorem 2.

3. Proofs of Theorems 3 and 4

It is a consequence of (1.4) that

$$\hat{R}(z) = (iz + A_0 + \hat{A}(z))^{-1}, \qquad \operatorname{Im} z \leqslant 0,$$

so that we obviously have

$$\begin{aligned} \hat{R}(z) &= (iz+1)^{-1}(1 - \hat{b}(z)(1 + \hat{b}(z))^{-1}), \\ \hat{b}(z) &= (A_0 - 1)(iz+1)^{-1} + \hat{A}(z)(iz+1)^{-1}, \qquad \operatorname{Im} z \leqslant 0 \end{aligned} \qquad (3.1)$$

From (1.22) and (3.1) we deduce that

$$\hat{b}(z) + 1 \neq 0, \qquad \operatorname{Im} z \leqslant 0 \tag{3.2}$$

We are going to show that if the assumptions of Theorem 3 hold then the function

$$b(t) = (A_0 - 1)\, e^{-t} + \int_0^t A(t-s)\, e^{-s}\, ds \tag{3.3}$$

satisfies the hypothesis of Theorem 1. We may without loss of generality assume that $a_1 \in C^2([0, 1])$ and then we define $b(t) = ab_1(t) + b_2(t)$, $t \in R^+$, where

$$b_1(t) = \int_0^t a_1(t-s)\, e^{-s}\, ds + a_1(0)\, e^{-t} - a_1'(0)\, e^{-t}, \qquad t \in R^+. \tag{3.4}$$

Clearly b_1 is nonnegative, nonincreasing and convex and

$$\int_0^t b_1(s)\, ds = O\left(\int_0^t a_1(s)\, ds\right),$$

$$\left(\int_0^t b_1(s)\, ds\right)^{-1} = O\left(\int_0^t a_1(s)\, ds\right)^{-1}$$

$$\text{and} \quad \left(\int_0^t sb_1(s)\, ds\right)^{-1} = O\left(\int_0^t sa_1(s)\, ds\right)^{-1} \qquad \text{as} \quad t \to \infty. \tag{3.5}$$

Since (2.6) holds and $\int_0^\infty a_1(t)\, dt = +\infty$ it follows from (3.3) that we also have

$$\limsup_{z \to 0\, \operatorname{Im} z < 0} |\hat{b}_2(z)|/|\hat{b}_1(z)| < 1, \tag{3.6}$$

and then there is no need to establish the first part of (1.11) since it was only used to prove (2.6).

From (1.21), (3.3) and (3.4) we conclude that $b_2 \in AC_{\text{loc}}(0, \infty)$, $\lim_{t\to\infty} b_2^{(j)}(t) = 0$, $j = 0, 1$. Write $a_2 = k_1 + k_2$, where $k_1 \in AC_{\text{loc}}(0, \infty)$ has compact support in R^+, $\int_t^\infty |k_1'(s)|\, ds \in L^1(R^+)$ and $k_2 \in C^2([0, 1])$. Then we have

$$b_2'(t) = -(A_0 - 1 - aa_1(0) + aa_1'(0))\, e^{-t} + k_1(t) - \int_0^t e^{-(t-s)} k_1(s)\, ds$$

$$+ k_2(0)\, e^{-t} + \int_0^t k_2'(t-s)\, e^{-s}\, ds.$$

Now it is straightforward to check that if $b_4(t) = \int_t^\infty \operatorname{var}(b_2' \colon [s, \infty))\, ds$, then $b_4 \in L^1_{\mathrm{loc}}(R^+)$ and

$$\int_0^t s^j b_4(s)\, ds = O\left(\int_0^t s^j a_4(s)\, ds\right) \qquad \text{as} \quad t \to \infty, j = 0, 1. \tag{3.7}$$

Thus we can apply Theorem 1 to the function b, see (3.2) and (3.4)–(3.7).

Choosing the constant c in (2.31) and (2.37) to be $\frac{1}{2}$ we see that we can replace T by $2T$ on the right-hand side in (1.18) and (1.19). Thus it follows from (3.1), (3.5) and Theorem 1 that the assertion of Theorem 3 is true.

To prove Theorem 4 we proceed in the same manner as above, the only difference being that now we define b_1 to be

$$b_1(t) = \int_0^t a_1(t-s)\, e^{-s}\, ds + a_1(0)\, e^{-t} - a_1'(0)\, e^{-t}$$
$$+ a_1''(0)\, e^{-t}, \qquad t \in R^+.$$

This completes the proof of Theorems 3 and 4.

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